Symbolic solutions of algebraic ordinary differential equations

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Introduction

Problem
Given: autonomous AODE $F(y, y') = 0$
Compute a rational/radical/…solution y

Definitions

- Ring of differential polynomials: $\mathbb{K}(x)\{y\} = \mathbb{K}(x)[y, y', y'', \dots]$
- AODE: $F(x, y, y', \dots, y^{(n)}) = 0$ with $F \in \mathbb{K}(x)\{y\}$ polynomial in x
- Autonomous: x does not appear in coefficients
- Curve: $\mathcal{C} = \{(a, b) \in \mathbb{K}^2 \mid f(a, b) = 0\}$
- Parametrizations: $\mathcal{P}(t) = (r(t), s(t))$ such that $f(r(t), s(t)) = 0$ and $\mathcal{P}(t) \notin \mathbb{K}^2$
- Proper: $\mathcal{P}(t) : \mathbb{A} \to \mathcal{C}$ is birational

Rational Solutions

Algorithm: Feng and Gao [1]
Given: AODE $F(y, y') = 0$

- Compute a proper rational parametrization $P(t) = (r(t), s(t))$
- Compute $A = \frac{s(t)}{r'(t)}$
- If $A \in \mathbb{Q}$ or $A = a(b + t)^2$ then $y(x) = r(A(x - c))$ or $y(x) = r(\frac{-1+ab(x-c)}{a(x-c)})$
- Otherwise there is no rational solution

Note: (y, y') is a proper rational parametrization

Generalizations

- Non-autonomous AODEs $F(x, y, y') = 0$ [9, 10, 11]
- Higher order AODEs [5, 6]
- Transformations [7, 8]

Radical Solutions

Procedure
Given: AODE $F(y, y') = 0$

- Compute a parametrization $\mathcal{P}(t) = (r(t), s(t))$ of $F(y, z) = 0$
- Assume $\mathcal{P}(t) = (y(g(t)), y'(g(t)))$ for some g
- Compute $A_{\mathcal{P}}(t) := \frac{s(t)}{r'(t)}$
- Compute $g(t)$ (integration) and $g^{-1}(t)$ (inverse function)
- General solution $y(x) = r(g^{-1}(x + c)$

Radical Parametrizations
$\mathcal{P}(t) = (r(t), s(t))$ such that $F(r(t), s(t)) = 0$ and $r(t), s(t)$ are in some radical extension field of $\mathbb{K}(t)$. For a precise definition and further information see [13, 14, 4].

Theorem
Let $\mathcal{P}(t) = (r(t), s(t))$ be a radical parametrization of the curve $F(y, z) = 0$ and assume $A_{\mathcal{P}}(t) = a(b + t)^n$ for some $n \in \mathbb{Q} \setminus \{1\}$.
Then $r(h(t))$, with $h(t) = -b + (-(n-1)a(t + c))^{\frac{1}{1-n}}$, is a radical general solution of the AODE $F(y, y') = 0$.

Theorem
Let $\mathcal{P}(t) = (r(t), s(t))$ be a radical parametrization of the curve $F(y, z) = 0$. Assume $A_{\mathcal{P}}(t) = \frac{at^n}{b+t^m}$ for some $a, b \in \mathbb{Q}$ and $m, n \in \mathbb{Q}$ with $m \neq n - 1$ and $n \neq 1$.
Then the AODE $F(y, y') = 0$ has a radical solution if the function

$$g(t) = \frac{1}{a} t^{1-n} \left(\frac{b}{1-n} + \frac{t^m}{1+m-n} \right) \qquad (1)$$

has a radical inverse $h(x)$. A general solution of the AODE is then $r(h(x + c))$.

Theorem
Assume $1 - n = \frac{z_1}{d_1}$ and $m - n + 1 = \frac{z_2}{d_2}$ with $z_1, z_2 \in \mathbb{Z}$, $d_1, d_2 \in \mathbb{N}$ such that $\gcd(z_1, d_1) = \gcd(z_2, d_2) = 1$. Let $\bar{n} = \frac{(1-n)d_1d_2}{d}$, $\bar{m} = \frac{(m-n+1)d_1d_2}{d}$ and $d = \gcd(z_1d_2, z_2d_1)$.
The function $g(t)$ of (1) has an inverse expressible by radicals if

- $b = 0$ or
- $\pm(\bar{m}, \bar{n}) \in \mathbb{N}^2$ and $\max(|\bar{m}|, |\bar{n}|) \leq 4$.
- $\pm(-\bar{m}, \bar{n}) \in \mathbb{N}^2$ and $|\bar{m}| + |\bar{n}| \leq 4$.

It has no inverse expressible by radicals in the cases

- $\bar{m}, \bar{n} \in \mathbb{N}$ and $\max(\bar{m}, \bar{n}) > 4$,
- $-\bar{m}, -\bar{n} \in \mathbb{N}$ and $\max(|\bar{m}|, |\bar{n}|) > 4$.

Example & Conclusion

Example
AODE: $y'^6 + 49yy'^2 - 7 = 0$

$$\mathcal{P}(t) = \left(-\frac{-7 + t^6}{49t^2}, t \right), \qquad A_{\mathcal{P}}(t) = -\frac{49t^4}{14 + 4t^6}$$

$$g(t) = \frac{2}{21t^3} - \frac{4t^3}{147}, \qquad g^{-1}(t) = \frac{1}{2} \left(-147t - \sqrt{7}\sqrt{32 + 3087t^2} \right)^{1/3}$$

$$y(x) = -\frac{4 \left(-7 + \frac{1}{64} \left(-147(c + x) - \sqrt{7}\sqrt{32 + 3087(c + x)^2} \right)^2 \right)}{49 \left(-147(c + x) - \sqrt{7}\sqrt{32 + 3087(c + x)^2} \right)^{2/3}}.$$

Conclusion

- General procedure for autonomous AODEs
- Radical solutions for some classes
- Also possible for non-radical solutions
- Solves AODEs not solveable by current CAS
- Generalizes rational case
- Future research: Complete decision algorithm

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